A TECHNIQUE FOR DETERMINING CLOSURE IN SEMANTIC TABLEAUX

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Introduction

The author considers the model-theoretic character of proofs and disproofs by means of attempted counterexample constructions, distinguishes this proof format from formal derivations, then contrasts two approaches to semantic tableaux proposed by Beth and Lambert-van Fraassen. It is noted that Beth's original approach has not as yet been provided with a precisely formulated rule of closure for detecting tableau sequences terminating in contradiction. To remedy this deficiency, a technique is proposed to clarify tableau operations.

A Technique for Determining Closure in Semantic Tableaux

By the mid-1950's, contributions to natural deduction methods exhibited a pronounced model-theoretic character. Building on the work of Herbrand and Gentzen, Beth, Hintikka, Schütte and others developed techniques for finding a proof of a first-order formula by demonstrating the impossibility of constructing a counterexample.[1]

1. (For works by Beth, see references.) *Inter alia*: Jaakko Hintikka, "Distributive Normal Forms in the Calculus of Predicates," *Acta Philosophica Fennica*, Fasc. VI, Helsinki, 1953; "Form and Content in Quantification Theory," *Acta Philosophica Fennica*, Fasc. VIII, Helsinki, 1955; "Distributive Normal Forms in First-Order Logic," in J. N. Crossley and M. A. E. Dummett, (Eds.), *Formal Systems and Recursive Functions*, Amsterdam: North Holland Publishing Co., 1965, pp. 47-90. Kurt Schütte, "Schlussweisen-Kalküle der Prädikatenlogik," *Mathematische Annalen 122*, 1950; "Ein System des verknüpfenden Schliessens," *Archiv für mathematische Logik und Grundlagenforschung*, 2, 1956. S. C. Kleene, *Introduction to Metamathematics*, Princeton, N. J.: D. Van Nostrand, 1952.

Descriptions of a counterexample may literally be thought of as models in which a falsifying instance of the formula in question is systematically forced. When a systematic exploration of the conditions such a counterexample would have to satisfy necessarily terminates in contradiction, the original formula is proved.

The idea of conceiving of proofs and disproofs as attempted model constructions may be one of the most philosophically interesting. Two of the strongest techniques of argumentation in philosophy share aspects of this approach: (i) *reductio ad absurdum*, applied to show that a position, often denying the one endorsed, results in contradiction,[2] and (ii) *self-referential argumentation*, serving to demonstrate that positions conflicting with one's own are self-defeating.[3]

As yet, there has been little explicit use made in philosophical argumentation of recently developed techniques of proof and disproof through counterexample construction. One of the reasons for this may perhaps be that the best known of these techniques, Beth's method of semantic tableaux, has still to be formulated in a clear manner which facilitates its use. It is one of the purposes of this paper to contribute to this clarification.

An approach to proofs and disproofs by means of a systematic construction of a possible counterexample differs in several ways from formal derivations. A formal derivation of X from premisses A_1, A_2, . . ., is found when successive applications of available rules of inference yield the conclusion X. Alternatively, a proof of X is found if it can be shown that it is not possible without contradiction to construct a

2. See, e.g., Gilbert Ryle, "Proofs in Philosophy," *Revue internationale de philosophie VIII*, 1954; "Philosophical Arguments," in A. J. Ayer, (Ed.), *Logical Positivism*, Glencoe, Illinois: The Free Press 1959, pp. 327-344; John Passmore, *Philosophical Reas ning*, London: G. Duckworth, 1961; Warren J. Hockenos, *An examination of Reductio ad Absurdum and Argumentum ad Hominem Arguments in the Philosophies of Gilbert Ryle and Henry W. Johnstone, Jr.*, Boston University, Ph.D. di sertation, 1968.

3. See, for example, Frederic Brenton Fitch, "Self-Reference in Philosophy," *Mind 55*, 1946; Henry W. Johnstone, Jr., *Philosophy and Argument*, Philadelphia, Pa.: Pennsylvania State University Press, 1959; John Passmore, *Philosophical Rea oning*, London: Gerald Duckworth and Co., 1961, Chap. 4; J. L. Mackie, "Self-Refutation – A Formal Analysis," *The Philosophical Quarterly 14*, 1964; Hockenos, 1968 (see preceding note); and the author's "The Idea of a Metalogic of Reference," *Methodology and Science 9*, 1976; "Self-Reference, Phenomenology, and Philosophy of Science", *Methodology and Science, 13*, 1980; and "Referential Consistency as a Criterion of Meaning", *Synthèse 52*, 1982.

falsifying instance such that both $\{A_1, A_2, \ldots\}$ and $-X$ are affirmed. On the other hand, if one wished to show that $A_1, A_2, \ldots \nvdash X$, it would be difficult, to say the least, to investigate all formal derivations starting from the premisses $A_1, A_2, \ldots,$ and thereby to determine that none leads to the conclusion X. Alternatively, it would be enough to identify an appropriate counterexample to $A_1, A_2, \ldots \vdash X$. In short, from the standpoint of formal derivability, a derivation of X from $A_1, A_2, \ldots$ may be attempted: if a derivation is found, then $\vdash X$; if no such derivation exists, then $\nvdash X$. Alternatively, we may try to construct a counterexample; if none can be constructed without contradiction, then $\vdash X$; if a counterexample can be produced, then $\nvdash X$.

There are two principal but contrasting approaches to proofs and disproofs by means of attempted constructions of counterexamples. (1) Beth's original method of semantic tableaux (Beth 1955, 1959, 1962, and passim) enables one to explore exhaustively and in a systematic manner all semantical conditions which must be satisfied in order for a counterexample to be possible. If a tableau reveals that a counterexample is logically impossible, then the original formula is known to be a theorem. If, on the other hand, a counterexample is constructed, the formula is known not to be a theorem.

(2) Lambert and van Fraassen (1972) have proposed a technique, based on Beth (1962), in which a group of rules is formulated so as explicitly to reduce formulas to disjunctive normal form, and thereby make it possible to decide whether a formula is or is not a theorem.

The two approaches yield, in practice, identical results. The methods are effective for the propositional calculus: if a propositional calculus formula is a theorem, its semantic tableau will show the impossibility of constructing a counterexample. If the formula is not a theorem, each method will systematically yield a counterexample. For the first-order predicate calculus, semantic tableaux will not, as one would expect, serve to detect without fail all invalid inference forms or non-theorems.

The two approaches are virtually the opposite of one another in terms of procedure. In Beth's presentation of his method of semantic tableaux, if an inference is analyzed, the formula(s) constituting the premisse(s) is (are) placed on the left (True) side of the tableau, and the formula comprising the conclusion is placed on the right (False) side. If a formula, rather than an inference, is analyzed, the formula is placed on the right side of the tableau. In either of these cases, applying Beth's tableau rules, if a contradiction is reached, then the contradiction indicates (a) that the semantical conditions which a possible counterexample must satisfy are mutually incompatible, hence a counterexample is impossible; and therefore (b) that the inference or formula is valid or is a theorem. If a contradiction is not reached, a propositional calculus inference or theorem is shown to be invalid or to be a non-theorem. In the case of the predicate calculus, if a contradiction is reached, validity or theoremhood is assured; if no contradiction is reached, and the tableau sequence cannot be continued, then the inference or formula is shown to be invalid or to be a non-theorem. However, in those cases when the predicate calculus tableau sequence cannot be terminated, one is unable to determine using semantic tableaux whether the inference or formula is or is not an invalid inference or is or is not a theorem.

In the Lambert-van Fraassen approach, if one wishes to determine for a given formula whether it is a theorem, then the tableau sequence is initiated with the negation of that formula. Applying the Lambert-van Fraassen tableau rules, the negated formula is reduced to disjunctive normal form. If the resulting disjunction is such that each and every of its disjuncts is a conjunction of one or more pairs of contradictory expressions, then the method leads to the conclusion that the original, unnegated formula is a theorem. The contradictions expressed by all the disjuncts reveal here, as in Beth's approach, that a counterexample cannot be constructed. Similarly, to determine whether a certain formula is a non-theorem, one begins the tableau sequence with that formula, follows the tableau rules, and, if a contradiction is reached, concludes that the formula is logically false.

In short, the two approaches to semantic tableaux may be contrasted as follows:

Beth	Lambert-van Fraassen
To show that a formula A is a theorem:	
Place A in False Column.	Initiate tableau sequence with $-A$.

If the tableau sequence terminates with a contradiction, *A* cannot consistently be falsified, hence *A* is a theorem.	If the tableau sequence terminates with a contradiction, *A* is a theorem.

If the tableau sequence does not terminate in a contradiction, but cannot be continued, then a counterexample has been identified, hence *A* is not a theorem.

To show that a formula A is not a theorem:

Place *A* in False Column	Initiate tableau sequence with *A*.
If the tableau sequence terminates *without* a contradiction, *A* can be falsified, hence *A* is not a theorem.	If the tableau sequence terminates *with* a contradiction, *A* is not a theorem (although $-A$ is shown to be).

The original approach to semantic tableaux of Beth and the later version due to Lambert and van Fraassen can both be formulated algorithmically in terms of their respective sets of tableau rules: i.e., both approaches are programmable on a logic machine. The principal difference between the two approaches, which results in the contrasting proof strategies we have noted, lies of course in differing formulations of the tableau rules. (For a statement of these rules, the reader is referred to Beth (1955, 1959) and van Fraassen (1972).)

Beth's original method has the decided advantage of more closely approximating natural reasoning patterns. The technique enables one to analyze explicitly and systematically the semantical conditions which a falsifying instance must satisfy. Where the Lambert-van Fraassen approach may be thought of as a set of rules to reduce formulas to disjunctive normal form, Beth's method offers a procedure for undertaking what might be termed a presuppositional analysis of the semantical structure of formulas.

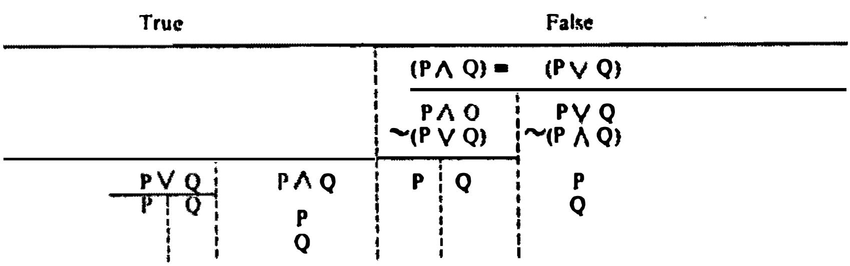
Unfortunately, Beth's method requires some procedure to determine closure, the statement of which has suffered from imprecision and ambiguity. A technique for determining closure is needed to make clear when a tableau sequence terminates in contradiction. We recall that a tableau sequence will either (i) terminate in contradiction, (ii) terminate not in contradiction but because it cannot be continued, or (iii) not terminate at all. In the first case, the tableau sequence reveals the logical impossibility of constructing a suitable counterexample. In the second case, a suitable counterexample is described. In the third case, the tableau fails to reveal whether a formula is or is not a logical truth. A technique for determining closure is needed to decide between the case in which a counterexample is not constructible and the case in which a counterexample has been described. For these cases are by no means always readily distinguished in complex tableaux.

In an effort to gain the needed clarity, the following formulation of a rule of closure was proposed: "A contradiction is reached when (i) there are no subcolumns and a symbol appears on both sides of the tableau, or (ii) there are subcolumns, and for each subcolumn on one side of the tableau there are symbols occurring in it such that at least one of them occurs in every corresponding subcolumn on the other side of the tableau."[4]

Determining closure is straightforward when neither side of a tableau divides into subcolumns. Difficulties arise only in connection with the existence of subcolumns on one side or both sides of a tableau. The tableaux with which we are here concerned are of this latter variety. To give some idea of the difficulties these tableaux pose we will consider several examples:

The above rule questionably covers certain cases (see, e.g., tableau (i), below), and is ambiguous when applied to certain others (tableaux (ii), (iii), (iv)).

Tableau (i). To prove that $(P \wedge Q) \equiv (P \vee Q)$ is not a theorem.



Comment: The tableau sequence appears to meet condition (ii) of th rule, yet, clearly, the formula in question is not a the rem. Some revision of the rule of closure in question is needed.

4. Brody, 1973, p. 128.

Tableau (ii). To prove that the inference $(x)(Fx \rightarrow Gx)$, $(\exists x)(Fx \wedge Hx) \vdash (\exists x)(Gx \wedge Hx)$ is valid.

True	False
$(x)(Fx \rightarrow Gx)$ $(\exists x)(Fx \wedge Hx)$ $(x) \sim (Gx \wedge Hx)$ $Fa \wedge Ha$ Fa Ha $Fa \rightarrow Ga$	$(\exists x)(Gx \wedge Hx)$

True		False	
$\sim Fa$	Ga	Fa	
$\sim (Ga \wedge Ha)$	$\sim (Ga \wedge Ha)$	$Ga \wedge Ha$	$Ga \wedge Ha$
		Ga ¦ Ha	Ga ¦ Ha

Comment: The tableau sequence terminates in contradiction: a counterexample is impossible. However, Brody's formulation of the rule of closure is unclear for this case.

Tableau (iii). To prove that the inference $(x)(Fx \rightarrow (Gx \vee Hx))$, $(x)(Gx \rightarrow Ix)$, $(x) \sim Ix \vdash (x)(Fx \rightarrow Hx)$ is valid.

True	False
$(x)(Fx \rightarrow (Gx \vee Hx))$ $(x)(Gx \rightarrow Ix)$ $(x) \sim Ix$ $(\exists x) \sim (Fx \rightarrow Hx)$ $\sim (Fa \rightarrow Ha)$ Fa $Fa \rightarrow (Ga \vee Ha)$ $Ga \rightarrow Ia$ $\sim Ia$	$(x)(Fx \rightarrow Hx)$ $Fa \rightarrow Ha$ Ha Ia

True			False	
$\sim Fa$	$Ga \vee Ha$		Fa	
	Ga	Ha		
$\sim Ga$ ¦ Ia	$\sim Ga$ ¦ Ia	$\sim Ga$ ¦ Ia	Ga ¦	Ga ¦ ¦ Ga ¦

Comment: Same as for Tableau (ii).

Tableau (iv). To prove that the inference (x) (Fx → ~ Gx), (∃x) (Gx V Hx) ⊢ (∃x) (~ Fx V Hx) is valid.

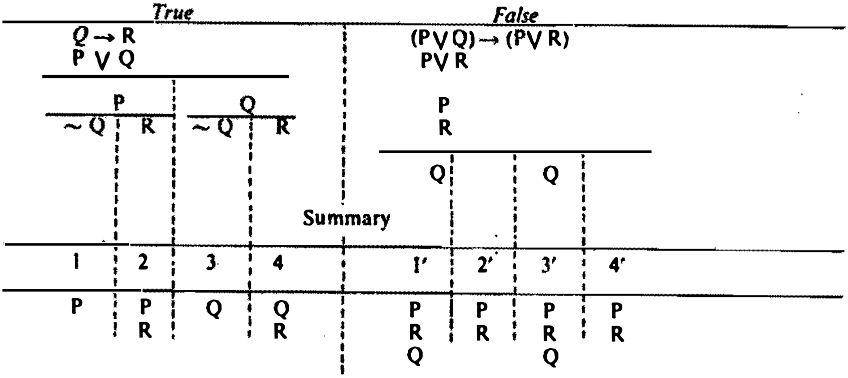
True	False
(x) (Fx → ~ Gx)	
(∃x) (Gx V Hx)	
	(∃x) (~ Fx V Hx)
(x) ~ (~ Fx V Hx)	
Ga V Ha	
Fa → ~ Ga	
~ (~ Fa V Ha)	
	~ Fa V Ha
	~ Fa
	Ha
Fa	
Ga ¦ Ha	
~ Fa ¦ ~ Ga ¦ ~ Fa ¦ ~ Ga	
	Fa ¦ Ga ¦ Fa ¦ Ga

Comment: Same as for Tableau (ii).

There has yet to be formulated a clearly stated technique for determining closure when, using Beth's approach, tableau sequences terminate in contradiction. That such a technique is needed should be evident from the presence of ambiguity in the few sample illustrations above. Semantic tableaux constitute a logically important technique, but one that is of little value if it is uncertain what conclusion is to be drawn from a given tableau sequence.

To determine whether a tableau sequence that involves subcolumns terminates in contradiction, it is helpful to proceed as follows: First determine whether the True- and False- sides of the tableau contain the same number of subcolumns. If one side contains more subcolumns than the other, subdivide further the side with fewer subcolumns, until both sides contain the same number of subcolumns, and in corresponding positions. The subcolumns of the tableau will now be bilaterally symmetrical with respect to the vertical midline of the tableau. Now, reproduce, below the last line of the tableau sequence, a summary of all elementary expressions which have occurred earlier in the sequence, placing these expressions in all appropriate subcolumns, as illustrated.

Tableau (v). To prove that the inference Q → R ⊢ (P∨Q) → (P∨R) is valid.

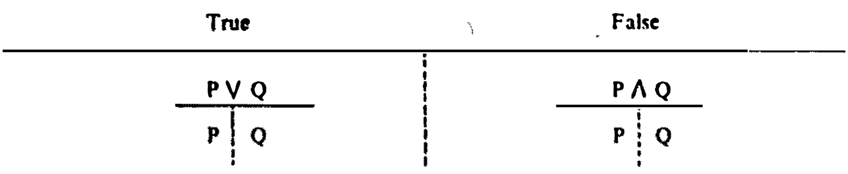


It will be noted that only elementary expressions are listed in the tableau summary: negations of elementary expressions are not summarized.

Having summarized in this fashion all tableau occurrences of elementary expressions by reproducing them in appropriate subcolumns, three cases may arise:

In the first, the subcolumns on the left side each contains only a single elementary expression. For example:

Tableau (vi). To prove that the inference P∨Q ⊢ P ∧ Q is invalid.



If this is the case in question, *a contradiction occurs in the tableau sequence if, and only if, each of the elementary expressions on the left can be paired with an identical expression in every subcolumn on the right.* (Rule A) In the example above, no contradiction is reached, since ‘P’ and ‘Q’ do not appear in both subcolumns on the right; hence the inference in question is invalid.

A second case is also possible when the subcolumns on the left side contain multiple elementary expressions. For example:

Tableau (vii). To prove that (P ∧ Q) ≡ (Q ∧ P) is a theorem.

True				False			
				(P ∧ Q ≡ (Q ∧ P)			
				P ∧ Q ~(Q ∧ P)		Q ∧ P ~(P ∧ Q)	
Q ∧ P Q P		P ∧ Q P Q					
				P	Q	Q	P
1	2	3	4	1′	2′	3′	4′
P Q	P Q	P Q	P Q	P	Q	Q	P

If this is the case in question, *a contradiction occurs in the tableau sequence if and only if every subcolumn on the right contains at least one of the expressions found within each corresponding subcolumn on the left.* (Rule B) Or, equivalently stated, and referring to the example: 'P', 'Q' appear in the leftmost subcolumn. One or the other of these appears in each corresponding subcolumn on the right. This is true for each of the other subcolumns on the left side of the vertical mainline; hence the equivalence (P ∧ Q) ≡ (Q ∧ P) is a theorem.

In more complex tableau sequences, both of the above cases will be encountered together. In such "mixed" tableaux, rule A is weakened as follows: Identify every subcolumn on the True-side, each of which contains only a single elementary expression. Those subcolumns on the False-side that correspond in position are termed 'related subcolumns'. Then, *a contradiction occurs in this portion of the tableau sequence if and only if each of the elementary expressions on the left can be paired with an identical expression in every related subcolumn on the right.* (Rule A')

For example, to analyze the expression (P ∧ Q) ≡ (P ∨ Q), which is clearly not a theorem:

Tableau (viii).

True		*False*	
		$(P \wedge Q) \equiv (P \vee Q)$	
		$P \wedge Q$ $\sim(P \vee Q)$	$P \vee Q$ $\sim(P \wedge Q)$
$P \vee Q$	$P \wedge Q$ P		
P | Q	Q		
		P | Q	P Q

Summary

1	2	3	4	1′	2′	3′	4′
P	Q	P Q	P Q	P	Q	P Q	P Q

Subcolumns 1 and 2 exhibit case 1; subcolumns 3 and 4 belong to case 2. Although a contradiction is partially shown by inspecting subcolumns 3 and 4 ('P' or 'Q' appears in corresponding subcolumns 3' and 4'), a contradiction is not revealed in checking subcolumns 1 and 2 ('P' does not occur in both related subcolumns 1' and 2', nor does 'Q' appear in both related subcolumns 1' and 2'.) Therefore, the expression in question, $(P \wedge Q) \equiv (P \vee Q)$, is not a theorem, since the tableau sequence does not terminate in contradiction.

A third, quasi-limiting case may also arise, one that is less frequently met with, and easily treated. It is possible that one or more subcolumns in the tableau summary may be empty. When an empty subcolumn occurs on one side of a tableau, it will be found that the corresponding subcolumn on the other side of the tableau is not empty, but rather effectively identifies a counterexample. To illustrate this:

Tableau (ix). To show that the inference P ≡ Q ⊢ P ∧ Q is invalid.

True				False			
P ≡ Q				P ∧ Q			
P Q		~ P ~ Q				P Q	
				P	Q	P	Q
1	2	3	4	1'	2'	3'	4'
P Q	P Q			P	Q	P Q	P Q

Subcolumns 3 and 4 are empty, but their corresponding subcolumns 3' and 4' are not empty, hence a counterexample *is* possible, and thus the inference in question is invalid. The counterexample is constructed by assigning to the sequent the values P=F and Q=F (the values indicated by subcolumns 3' and 4', which correspond to the empty subcolumns 3 and 4).

The tableau for ⊢ (P → Q) → ~ P illustrates the opposite asymmetry:

Tableau (x).

True			False	
P → Q P			(P → Q) → ~ P ~ P	
~ P	Q	Summary	P	
1	2		1'	2'
P	P Q		P	

Here, the counterexample is constructed by assigning to the expression, (P→Q) → ~ P, the values P=T and Q=T (the values indicated by subcolumn 2, which corresponds to the empty subcolumn 2').

Such considerations suggest a third rule of closure: *No subcolumns in* *the tableau summary are to be empty.* (Rule C) For, as we have observed, the occurrence of an empty subcolumn immediately identifies a counterexample.

The closure rules A, A', B, and C together establish a technique for determining closure in semantic tableaux. To be perfectly clear, the technique requires that the meta-rule, that all pertinent closure rules must be satisfied, itself must of course be in force.

To illustrate the proposed method for determining closure somewhat further, let us again look at tableau (iii), which provides a more complex example.

(x) (Fx → (Gx ∨ Hx)), (x) (Gx → Ix), (x) ∼Ix ⊢ (x) (Fx → Hx)

True	False
(x) (Fx → (Gx ∨ Hx) (x) (Gx → Ix) (x) ∼ Ix (∃x) ∼ (Fx → Hx) ∼ (Fa → Ha) Fa Fa → (Ga ∨ Ha) Ga → Ia ∼ Ia	(x) (Fx → Hx) Fa → Ha Ha Ia

∼ Fa		Ga ∨ Ha				Fa					
		Ga		Ha							
∼ Ga	Ia	∼ Ga	Ia	∼ Ga	Ia	Ga		Ga		Ga	
					Summary						
1	2	3	4	5	6	1'	2'	3'	4'	5'	6'
~~Fa~~	~~Fa~~ ~~Ia~~	Fa ~~Ga~~	Fa Ga ~~Ia~~	Fa ~~Ha~~	Fa ~~Ha~~ ~~Ia~~	Ha Ia ~~Fa~~ Ga	Ha ~~Ia~~ ~~Fa~~	Ha Ia ~~Ga~~	Ha ~~Ia~~	~~Ha~~ Ia Ga	~~Ha~~ ~~Ia~~

Each elementary expression is reproduced in all subcolumns in the tableau sequence, below its occurrence – i.e., in the example, 'Fa' is placed in all True-subcolumns 1-6 since 'Fa' occurred prior to the appearance of subcolumns in the tableau. Similarly, 'Ga' is written in subcolumns 3 and 4 since 'Ga' occurred above the subdivision of these subcolumns. 'Ia' is reiterated in subcolumns 2, 4, and 6. The process is continued for 'Ha' in subcolumns 5 and 6, 'Ha' and 'Ia' in False-subcolumns 1'-6', and 'Fa' in subcolumns 1' and 2'. 'Ga' is reiterated in 1', 3' and 5'.

Now, on the left side of the tableau, only subcolumn 1 contains a single elementary expression. Subcolumn 1', the related subcolumn on the right side, contains the same expression, satisfying the rule A'. Subcolumns 2-6 each contains one of the expressions listed in corresponding subcolumns 2'-6', satisfying rule B. Rule C is satisfied. Expressions with a slash through them identify those which conform to tableau rules A' and B, and which recur within corresponding subcolumns. The tableau summary for (iii) indicates clearly and graphically that the sequent in question is valid.

We may observe the following general result concerning the proposed technique for determining closure: If, and only if, in each non-empty subcolumn in the tableau summary at least one expression is slashed, then the tableau sequence terminates in contradiction, revealing that no counterexample is constructible.

At this point, the reader may wish explicit justification for the set of rules that has been introduced. To assuage any doubts concerning the effectiveness of these rules, the following informal proofs are given:

Each of the rules is to be applied after a tableau sequence has been summarized in the manner described. (With some practice, of course, this summary need only take place mentally.) Each of the rules describes conditions which must be satisfied if the tableau sequence is to terminate in contradiction. A proof that each rule succeeds in identifying such conditions therefore must show that the attempt to construct a counterexample becomes impossible because, for each alternative occurring on the True-side of the tableau, pairs of semantically incompatible, i.e., contradictory, propositions must be asserted.

Each of the rules focuses on the status of subcolumns that appear on the left side of the tableau summary.

Rule A refers to the case in which every subcolumn on the left each contains but a single elementary expression. We recall that each lefthand subcolumn expresses a (non-exclusive) semantical alternative. If a lefthand subcolumn contains the expression 'P', and 'P' occurs in all righthand subcolumns, this indicates that P is true in the given alternative, but that P is also false in all alternatives which appear on the righthand side of the tableau. When this situation occurs in connection with every alternative on the lefthand side, a counterexample cannot then be constructed without forcing a logical inconsistency.

Rule B refers to the case in which the summary subcolumns on the lefthand side of a tableau contain more than only a single expression. Each subcolumn expresses a semantical alternative in which all the expressions occurring in that subcolumn are true – i.e., the expressions comprise a conjunction which is true in that alternative. If a lefthand subcolumn contains the expressions 'P', 'Q', and one or the other of these occurs in the corresponding alternative on the righthand side, then clearly a contradiction, e.g., $P \wedge Q$, $\sim P$, is entailed. When this situation occurs in connection with every alternative of this conjunctive variety on the lefthand side, a counterexample cannot then be constructed without contradiction.

Rule A' refers to the so-called "mixed" case: some lefthand subcolumns contain only single expressions, some are conjunctions of two or more expressions. Some alternatives will thus involve conjunctions, whose status in the tableau is determined by attempting to match at least one conjunct with an expression in a corresponding false alternative. If Rule B is satisfied, then it is sufficient to match the single alternative expressions with identical expressions in their related false subcolumns. In effect, this demonstrates that every non-conjunctive alternative is contradicted by every false alternative that remains to be considered provided that Rule B is satisfied. All alternatives, which under Beth's method of semantical analysis must be true (if a counterexample is to be possible), are then shown to entail inconsistency.

Rule C, which states the condition that there are to be no empty subcolumns in a tableau summary, is easily justified. Only if rule C is satisfied, is it possible for tableau sequences to terminate in contradiction. The occurrence of an empty subcolumn will correspond to a non-empty subcolumn on the other side of the tableau and therefore identifies a counterexample which is constructible without contradiction. The empty subcolumn expresses the fact that no semantical conflict can obtain for the value(s) indicated by the corresponding non-empty subcolumn.

It should be evident that a non-empty subcolumn will always correspond to an empty one, since two empty corresponding subcolumns cannot appear in a tableau sequence. Corresponding subcolumns express the semantical conditions associated with each half of paired disjunctions. If two corresponding subcolumns were empty, this would be tantamount to requiring that there exist a disjunction without a first (or, a second) disjunct, which is impossible.

Rules A, A', B, and C define a systematic technique for determining closure: With a little practice, one is able to work efficiently and without ambiguity in the context of Beth's original method of semantic tableaux.

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